of the nuclear potential thereby inferred. By choosing values of r_0 and M consistent with low-energy scattering experiments and our present understanding of nuclear matter a value of α has been derived which yields theoretical nuclear evaporation spectra which fit the present low excitation energy experimental data very well. However, when more precise experimental data at excitation energy ~ 20 MeV become available α may be determined more precisely than is possible at the present time. The dependence of α as well as β on excitation energy may then be interpreted in terms of the properties of nuclear matter ($M^*(E)$ and $\eta(E)$) as higher excitation data become available.

At modest excitation energies of only 14 MeV maximum the effect of the shape of the nuclear potential is relatively unimportant and the theory may be tested by neglecting the expected change in nuclear potential with excitation energy. Neutron emission spectra at modest excitation energy have been computed *without reference to any particle-emission experiments* (that is, nuclear dimensions used in the theory are determined from elastic neutron scattering experiments and the nuclear level densities in a potential of these dimensions are determined entirely theoretically) and the result has been compared successfully to 14-MeV inelastic neutron scattering and (p,n) measurements. Experimental temperature and parameter fitting are not useful concepts when multiple particle emission is at all possible.[19,20] Their experimental determination will result in a possible decrease of "temperature" with excitation energy and a level density parameter an order of magnitude lower than the theoretical value.

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Observable Consequences of Anomalous Thresholds

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The Landau surface for the triangle diagram touches the physical region at three points. The consequences of this singlar-matrix element for several processes are studied. It is found that there should be peaks in cross sections at the edge of phase space. These peaks depend sensitively on the incident energy and are distinguished in this way from genuine resonances.

I. INTRODUCTION

THE existence of anomalous thresholds in certain Feynman amplitudes was first noticed by Karplus, Sommerfield, and Wichmann.[1] There has been much subsequent work, particularly by Landau[2] who gave the general conditions for the occurrence of many particle singularities and by Cutkosky[3] who showed how to calculate the discontinuities across their cuts.

In all of the current theoretical approaches to elementary particle physics one abandons many concepts of Lagrangian field theory, but, none the less, assumes that the singularities of the perturbation amplitude are preserved in the correct amplitude.[4,5] Thus, one cannot ignore anomalous thresholds and maintain any semblance of logic. It does not seem a particularly desirable situation, then, that there is no physical evidence that these are anything more than mathematical apparitions.

Landshoff and Trieman[6] and, more recently, Aaron[7] have suggested reactions in which effects of the anomalous threshold occurring in the triangle diagram might be seen. They were limited, however, either by competing diagrams for the same reaction, or by the large distance of the singularity from the physical region. We have found that these limitations can to some extent be removed by allowing two external particles at each vertex of a closed loop graph. Also, we find that under certain conditions, to be described later, the strength of the singularity may be enhanced.

We find that it is necessary to include at least one unstable particle among the internal particles. The imaginary part of the mass of this particle keeps the singularity from actually touching the boundary of the physical region. For a narrow resonance this might not be too serious. In fact, if the effect turns out to be observable, it might provide some information on the widths of such resonances.

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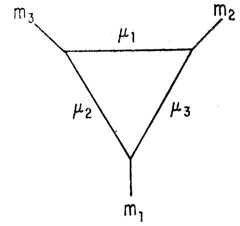

FIG. 1. The triangle diagram.

II. THE TRIANGLE SINGULARITY

Consider a reaction in which a particle of momentum $\mathbf{p}_1$ and mass m_1 decays into two particles of momenta $\mathbf{p}_2$ and $\mathbf{p}_3$ and masses m_2 and m_3, respectively. In the rest system of the first particle

$$m_1=(\mathbf{p}^2+m_2^2)^{1/2}+(\mathbf{p}^2+m_3^2)^{1/2}, \quad \text{where} \quad \mathbf{p}=\mathbf{p}_2=-\mathbf{p}_3.$$

$\mathbf{p}^2$ may be found by squaring twice:

$$\mathbf{p}^2=(m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2)/4m_1^2=\lambda(m)/4m_1^2.$$

In order that $\mathbf{p}^2$ be positive, $\lambda(m)$ must be positive. The function λ has four factors: $(m_1+m_2+m_3)$, $(m_1+m_2-m_3)$, $(m_1-m_2+m_3)$, and $(m_1-m_2-m_3)$. The last three correspond to the three possible disintegrations and the first to no physical process.

The surface $\lambda=0$ plotted in the space of the m^2 is a cone with vertex at the origin and tangent to each of the three planes $m_i^2=0$ along the line $m_j^2=m_k^2$. The physical region is external to the cone. In the space of the m_i, $\lambda=0$ is the surface of a triangular pyramid with apex at the origin, touching the coordinate planes along the coordinate planes along the lines $m_i=m_j$, which are also the edges of the pyramid. Again the region outside the pyramid is the physical region.

The (Landau) singularity surface of the triangle diagram (Fig. 1) is given parametrically by the equations

$$m_1^2=\mu_2^2+\mu_3^2-2\mu_2\mu_3(u_1^2-u_2^2-u_3^2)/2u_2u_3,$$
$$m_2^2=\mu_1^2+\mu_3^2-2\mu_1\mu_3(u_2^2-u_1^2-u_3^2)/2u_1u_3,$$
$$m_3^2=\mu_1^2+\mu_2^2-2\mu_1\mu_2(u_3^2-u_1^2-u_2^2)/2u_1u_2,$$

where the μ's are the internal masses, and the u's are three parameters. The function $\lambda(m)$ may be calculated in terms of the u's and μ's and is

$$\lambda(m)=\lambda(u)(\mu_1\mu_2/u_1u_2+\mu_1\mu_3/u_1u_3+\mu_2\mu_3/u_2u_3)^2.$$

Since the square is obviously positive, the sign of $\lambda(m)$ is the same as the sign of $\lambda(u)$. Expressed in terms of its factors $\lambda(u)$ is

$$\lambda(u)=-(u_1+u_2+u_3)(u_1+u_2-u_3)\times(u_1-u_2+u_3)(-u_1+u_2+u_3).$$

Since the u's must be the sides of a triangle for a physical sheet singularity, each of the four factors is positive and $\lambda(u)$ is negative or zero. When $\lambda(u)$ vanishes, the singularity surface touches the physical region. This happens when $u_1=u_2+u_3$, $u_2=u_3+u_1$, and when $u_3=u_1+u_2$. These values of the parameters correspond to the points:

$$\text{at } u_1=u_2+u_3, \quad m_1=|\mu_2-\mu_3|,\quad m_2=\mu_1+\mu_3,\quad m_3=\mu_1+\mu_2; \tag{1}$$

$$\text{at } u_2=u_3+u_1, \quad m_1=\mu_2+\mu_3,\quad m_2=|\mu_1-\mu_3|,\quad m_3=\mu_1+\mu_2; \tag{2}$$

$$\text{at } u_3=u_1+u_2, \quad m_1=\mu_2+\mu_3,\quad m_2=\mu_1+\mu_3,\quad m_3=|\mu_1-\mu_2|. \tag{3}$$

The reactions suggested by Landshoff and Treiman, and Aaron satisfy some of these conditions. For real points there are no minima for the separation of the singularity surface and the boundary of the physical region except at these three points. The three sets of conditions are equivalent and only the second will be treated with $\mu_3>\mu_1$. The point where the conditions are satisfied will be called P. Of the three equations the requirement that $m_2=\mu_3-\mu_1$ is the most difficult to satisfy. The other conditions can be satisfied by choosing a pair of light external particles at the vertices, with some internal kinetic energies. Trieman and Landshoff satisfied the difference condition at their q vertex by using a deuteron and two nucleons or by exploiting the small value of mass $(\Sigma)-$mass $(\Lambda)-$mass (π). The use of a bound state is not practical because a pole will almost certainly occur in the same process and will mask the anomalous threshold effect. Aaron suggested using a heavy unstable internal particle although he did not use it to satisfy the difference condition. The use of a heavy unstable particle for μ_3 is probably the most practical way of seeing an anomalous threshold. Another less practical possibility is the emission of a photon at the difference vertex.

The set of conditions (2) imply in addition to the Landau equations that each internal vertex a stability condition is just satisfied and the whole diagram is at the threshold. In terms of the analysis of singularities this causes the pinch to take place along a line in the plane of integration instead of at a point. For this reason, as will appear presently, the strength of the singularity is increased from logarithmic to the inverse square root type. The generalization of these conditions to more complicated diagrams is simple. To find a satisfactory example for a square diagram we would have to consider a six-particle final state.

The integral for the triangle diagram is

$$F_3=\int_0^1 d\alpha_1\int_0^1 d\alpha_2\int_0^1 d\alpha_3\,\delta(1-\alpha_1-\alpha_2-\alpha_3)/D,$$

where

$$D=m_1^2\alpha_2\alpha_3+m_2^2\alpha_1\alpha_3+m_3^2\alpha_1\alpha_2 -(\mu_1^2\alpha_1+\mu_2^2\alpha_2+\mu_3^2\alpha_3)(\alpha_1+\alpha_2+\alpha_3).$$

Close to the singularity the m's may be expressed as follows:

$$m_1=\mu_2+\mu_3+\delta_1,$$
$$m_2=\mu_3-\mu_1+\delta_2,$$
$$m_3=\mu_1+\mu_2+\delta_3,$$

where the δ's are small and it has been assumed that $\mu_3>\mu_1$ to lowest order in δ the denominator of the integral becomes

$$D=-(\mu_1\alpha_1-\mu_2\alpha_2-\mu_3\alpha_3)^2+2\alpha_1\alpha_2(\mu_1+\mu_2)\delta_3 +2\alpha_1\alpha_3(\mu_3-\mu_1)\alpha_2+2\alpha_2\alpha_3(\mu_2+\mu_3)\delta_1.$$

If the μ's are complex with small imaginary parts $\mu_i=\bar{\mu}_i+i\nu_i$ D may still be written as

$$D=-(\bar{\mu}_1\alpha_1-\bar{\mu}_2\alpha_2-\bar{\mu}_3\alpha_3)^2+\text{remainder},$$

where the remainder is small. Thus, D has a pair of roots near the line $\bar{\mu}_1\alpha_1-\bar{\mu}_2\alpha_2-\bar{\mu}_3\alpha_3=0$.

It is convenient to do first the α_3, then the α_2, and finally the α_1 integration. According to the preceding discussion, after doing the α_3 integration by means of the delta function, the denominator may be written as

$$D=A[\alpha_2-(\rho+\epsilon_1)][\alpha_2-(\rho+\epsilon_2)],$$

where A, ρ, and the ϵ's may depend on α_1. The α_2 integral may be carried out by elementary methods and gives

$$\frac{1}{A(\epsilon_1-\epsilon_2)}\ln\left[\frac{\alpha_2-\rho-\epsilon_1}{\alpha_2-\rho-\epsilon_2}\right]\Bigg|_0^{1-\alpha_1}.$$

The two roots $\rho+\epsilon_1$ and $\rho+\epsilon_2$ lie on opposite sides of the real α_2 axis since they must pinch the contour to produce a physical sheet singularity. For convenience, assume that $\rho+\epsilon_1$ has a positive imaginary part and that $\rho+\epsilon_2$ has a negative imaginary part. If α_2 passes the real part of these roots between the limits of the α_2 integration 0 and $1-\alpha_1$, then the phase of the logarithm will change by almost $2\pi i$, since the phase of the numerator increases by πi and that of the denominator decreases by πi. It is only this phase of the logarithm that contributes to the singularity. The integration becomes

$$F_3\approx 2\pi i\int\frac{d\alpha_1}{A(\epsilon_1-\epsilon_2)},$$

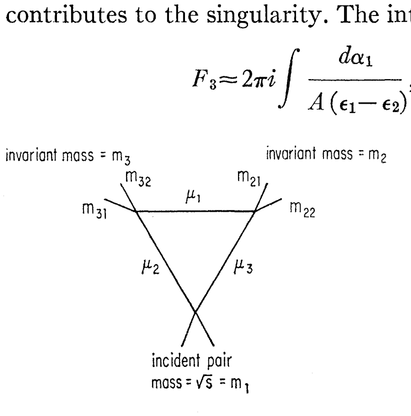

FIG. 2. The modified triangle diagram with two external particles attached at each corner.

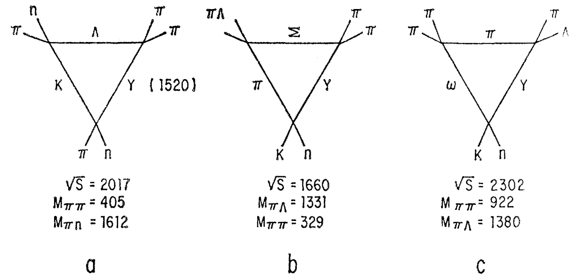

FIG. 3. Three reactions in which the anomalous threshold may be visible. The masses are given in MeV and no correction has been made for the shift in location of the peak due to the imaginary part of Y and ω. The Y is the 1520 MeV, strangeness -1 resonance, with a width of 15 MeV.

where

$$A(\epsilon_1-\epsilon_2)=[\lambda(m)\alpha_1^2+2B(m\mu)\alpha_1+C(m_1\mu)]^{1/2},$$
$$B(m\mu)=[-m_1^4+m_1^2m_2^2+m_1^2m_3^2+m_1^2\mu_2^2+m_1^2\mu_3^2 +m_2^2\mu_2^2+m_3^2\mu_3^2-m_2^2\mu_3^2-m_3^2\mu_2^2+2m_1^2\mu_1^2],$$
$$C(m\mu)=m_1^4+\mu_2^4+\mu_3^4-2m_1^2\mu_2^2-2m_1^2\mu_3^2-2\mu_2^2\mu_3^2,$$

and the integration is taken from $\alpha_1=\mu_2/(\mu_1+\mu_2)$ to $\mu_3/(\mu_1+\mu_3)$. At the point P each of the functions λ, B, and C vanishes but their derivatives with respect to the masses are finite. In the neighborhood of P, $A(\epsilon_1-\epsilon_2)$ may be written

$$A(\epsilon_1-\epsilon_2)=[(m_1-\mu_2-\mu_3)X +(m_2-\mu_3+\mu_1)Y+(m_3-\mu_1-\mu_2)Z]^{1/2},$$

where X, Y, and Z are the derivatives of $A(\epsilon_1-\epsilon_2)$ with respect to m_1, m_2, and m_3 evaluated at P. Next it is convenient to write each of the differences in polar form and incorporate the phase in the coefficients X, Y, and Z. Thus, $m_1-\mu_2-\mu_3=r_1e^{i\alpha}$, etc. With the modification, $A(\epsilon_1-\epsilon_2)$ becomes

$$A(\epsilon_1-\epsilon_2)=(X'r_1+Y'r_2+Z'r_3)^{1/2},$$

where the primes indicate that the phase factor has been absorbed. Finally, it is useful to introduce $r=(r_1^2+r_2^2+r_3^2)^{1/2}$ and write $r_i=r\cos\theta_i$. The final form of $A(\epsilon_1-\epsilon_2)$ is

$$A(\epsilon_1-\epsilon_2)=r^{1/2}(X''+Y''+Z'')^{1/2},$$

the double primes indicate the inclusion of the cosines. It is not necessary to do the integral to determine the behavior of the amplitude since after the $r^{1/2}$ is factored out the integral is well behaved and elementary. At the present stage of knowledge about anomalous thresholds there is no point in computing the integral. It describes some fine structure of the effect we wish to study and as the gross structure is unclear we ignore the fine structure for the time being. Our matrix element is then $r^{-1/2}$ times some scale factor.

To study the consequences of the $r^{-1/2}$ singularity in the matrix element it will be necessary to find a physical process that satisfies our conditions (2). In place of a

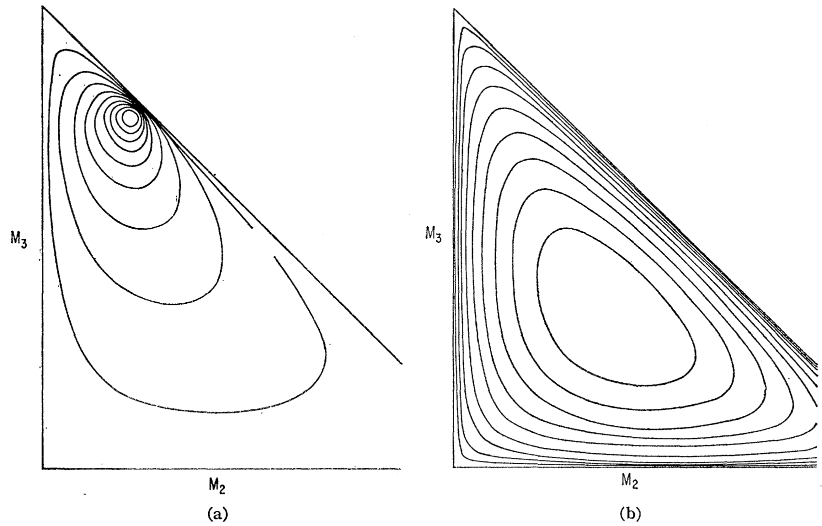

FIG. 4. Plots showing contours of events for the reaction $\pi+N\rightarrow(\pi n)+(\pi\pi)$ at 2017 MeV total center-of-mass energy. In Fig. 4(a) it is assumed that the diagram of Fig. 3(a) is the dominant process and in Fig. 4(b) the matrix element is taken equal to 1. In both plots, the contour interval is 10% of the maximum value. M_2 is the invariant mass of the π pair and M_3 is the invariant mass of the πn system.

single line having mass m_1 we shall attach the incident pair of particles having center-of-mass energy $s^{1/2}=m_1$. At each of the other two vertices we attach a pair of particles having center-of-mass energies m_2 and m_3, respectively. The masses of the constituents of these pairs are (m_{21},m_{22}) and (m_{31},m_{32}). The resulting diagram is shown in Fig. 2.

We calculate the relative probability of events with given $s^{1/2}$, m_2, and m_3 as a consequence of the $r^{1/2}$ matrix element. This probability is given by (phase space) $\times|r^{-1/2}|^2$. The phase space Γ is given by

$$\Gamma=\frac{\pi^3}{8m_2{}^2m_3{}^2s}\times[\lambda(s^{1/2},m_2,m_3)\lambda(m_2,m_{21},m_{22})\lambda(m_3,m_{31},m_{32})]^{1/2}.$$

Effectively only the first factor λ is relevant in the region that is of interest as the second two will be chosen slowly varying. Thus, the anomalous threshold produces a density of points ρ in a Dalitz plot in the neighborhood of P

$$\begin{aligned}\rho&=\lambda^{1/2}(s^{1/2},m_2,m_3)/r\\&=[|s^{1/2}-\mu_2-\mu_3|^2+|m_2-\mu_3+\mu_1|^2+|m_3-\mu_1-\mu_2|^2]^{-1/2}\\&\quad\times[s^2+m_2{}^4+m_3{}^4-2sm_2{}^2-2sm_3{}^2-2m_2{}^2m_3{}^2]^{1/2}.\end{aligned}$$

To see clearly what is happening several further approximations are useful. The second term may be factored as indicated earlier and only the factor $s^{1/2}-m_2-m_3$ that vanishes near P is important. In the first term some of the μ's are complex since we have included unstable particles. Let us call the contribution of the imaginary parts I^2, ρ becomes

$$\begin{aligned}\rho=[(s^{1/2}-\bar{\mu}_2-\bar{\mu}_3)^2+(m_2-\bar{\mu}_3+\bar{\mu}_1)^2\\+(m_3-\bar{\mu}_1-\bar{\mu}_2)^2+I^2]^{-1/2}\times(s^{1/2}-m_2-m_3)^{1/2},\end{aligned}$$

where I is the sum of the squares of the imaginary part of the masses. The maximum of ρ takes place at

$$s^{1/2}=\mu_2+\mu_3+I/\sqrt{3},\quad m_2=\mu_3-\mu_1-I/\sqrt{3},\quad m_3=\mu_1+\mu_2-I/\sqrt{3},$$

which is within the physical region.

III. SOME EXAMPLES AND CONCLUSIONS

In Figs. 3(a), (b), and (c) we show three processes that may exhibit our effect. As has already been pointed out[6] the effect in question occurs in a narrow range of the variables $s^{1/2}$, m_2, and m_3. The "width" of the peak is to be measured in units of I. On the other hand, the height of the peak depends on $I^{-1/2}$; hence, the effect is not easy to see. We have no way of estimating how significant the background is relative to the anomalous threshold. It also must be assumed that all the scattering amplitudes occurring at the corners of our triangle are large.

In Fig. 4 plots have been made of ρ for the first of the three reactions under the assumption that the triangle singularity is dominant and also with unit matrix element. It should be noted that the peak of triangle singularity comes at about 30–40% of the maximum of the pure phase-space plot. The peak for the triangle singularity falls off more slowly as $s^{1/2}$ is increased above the exact value for the singularity than if it decreases below this value.